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COMPLEX HYPERSURFACES OF A COMPLEX PROJECTIVE SPACE

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1. Statement of results

Let M be a compact complex hypersurface of the complex projective space $P_{n+1}(C)$. Then by a well known theorem of Chow, M is algebraic. We shall prove the following theorems.

Theorem 1. *Let M be a compact complex hypersurface of the complex projective space $P_{n+1}(C)$, and suppose that the Euler-Poincaré characteristic $\chi(M)$ of M is $n+1$. Then*

(1) *M is a complex hyperplane $P_n(C)$ if n is even.*

(2) *M is either a complex hyperplane $P_n(C)$ or a complex hyperquadric in $P_{n+1}(C)$ if n is odd.*

Theorem 2. *Let M be a complete complex hypersurface of the complex projective space $P_{n+1}(C)$. If every holomorphic sectional curvature of M is greater than $1/2$ with respect to the metric induced from the Fubini-Study metric of $P_{n+1}(C)$, then M is a complex hyperplane $P_n(C)$.*

It should be remarked that the referee of this paper has made the following conjecture stronger than Theorem 2: Let M be a complete complex hypersurface of the complex projective space $P_{n+1}(C)$. If M admits a Kaehler metric with respect to which M is of holomorphic pinching greater than $1/2$, then M is a complex hyperplane $P_n(C)$.

Theorem 3. *Let M be a compact complex hypersurface of the complex projective space $P_{n+1}(C)$. If every holomorphic sectional curvature of M is positive with respect to the metric induced from the Fubini-Study metric of $P_{n+1}(C)$, then M is either a complex hyperplane $P_n(C)$ or a complex hyperquadric in $P_{n+1}(C)$.*

2. Proof of Theorem 1

Let h be the generator of $H^2(P_{n+1}(C), Z)$ corresponding to the divisor class of a hyperplane $P_n(C)$. Then the total Chern class $c(P_{n+1}(C))$ of $P_{n+1}(C)$ is given by

$$c(P_{n+1}(C)) = (1 + h)^{n+2} .$$

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Let $j: M \to P_{n+1}(C)$ be the imbedding, ν the normal bundle of $j(M)$ in $P_{n+1}(C)$, and d the degree of the algebraic manifold M. Then the total Chern class $c(\nu)$ of ν is given by

$$c(\nu) = 1 + d\tilde{h} ,$$

where $\tilde{h}$ is the image of h under the homomorphism $j^*: H^2(P_{n+1}(C), Z) \to H^2(M, Z)$ induced by the imbedding $j: M \to P_{n+1}(C)$. Since $j^*T(P_{n+1}(C)) = T(M) \oplus \nu$ (Whitney sum), we have

$$j^*c(P_{n+1}(C)) = c(M)\cdot c(\nu) .$$

Let $c_i(M)$ be the i-th Chern class of M. Then we have

$$(1 + \tilde{h})^{n+2} = [1 + c_1(M) + \cdots + c_n(M)]\cdot(1 + d\tilde{h}) ,$$

which implies that

$$c_n(M) = [(1 - d)^{n+2} - 1 + (n + 2)d]\tilde{h}^n/d^2 .$$

Taking the values of both sides on the fundamental cycle of M, we have

$$\chi(M) = [(1 - d)^{n+2} - 1 + (n + 2)d]/d .$$

Since $\chi(M) = n + 1$, we have $(1 - d)[(1 - d)^{n+1} - 1] = 0$.

3. Proofs of Theorems 2 and 3

Let M be a complete complex hypersurface of $P_{n+1}(C)$ with the induced metric $g = 2\Sigma g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ and the fundamental 2-form $\Phi = \frac{2}{\sqrt{-1}}\Sigma g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$. Since every holomorphic sectional curvature is greater than $1/2$, M is compact. The first Chern class $c_1(M)$ of M is represented by the closed 2-form

$$\gamma = \frac{1}{2\pi\sqrt{-1}}\Sigma R_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta ,$$

where $S = 2\Sigma R_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta$ denotes the Ricci tensor of M. We denote $[\Phi]$ and $[\gamma]$ to be the cohomology classes represented by Φ and γ respectively, so that $c_1(M) = [\gamma]$.

The first Chern classes $c_1(P_{n+1}(C))$ and $c_1(M)$ are given by

$$c_1(P_{n+1}(C)) = (n + 2)h ,$$

$$c_1(M) = (n - d + 2)\tilde{h} . \tag{1}$$

Let Ψ be the fundamental 2-form of $P_{n+1}(C)$ so that

$$c_1(P_{n+1}(C)) = \frac{n + 2}{8\pi}[\Psi] .$$

These, together with the fact that $\Phi = j^*\Psi$, imply

$$[\Phi] = 8\pi\tilde{h} \ . \tag{2}$$

Let A be the tensor field of type $(1, 1)$ associated with the second fundamental form of the imbedding, J the complex structure tensor of M, and $e_1, \cdots, e_n, Je_1, \cdots, Je_n$ an orthonormal basis of $T_x(M)$ with respect to which the matrix of A is of the form

$$\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & 0 & \\ & & \lambda_n & & & \\ & & & -\lambda_1 & & \\ & 0 & & & \ddots & \\ & & & & & -\lambda_n \end{pmatrix} .$$

Let R be the curvature tensor of M. Then the equation of Gauss is

$$\begin{aligned} g(R(X, Y)Z, W) = {} & g(AX, W)g(AY, Z) - g(AX, Z)g(AY, W) \\ & + g(JAX, W)g(JAY, Z) - g(JAX, Z)g(JAY, W) \\ & + \frac{1}{4}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \\ & \qquad + g(JX, W)g(JY, Z) - g(JX, Z)g(JY, W) \\ & \qquad + 2g(X, JY)g(JZ, W)] , \end{aligned}$$

from which it follows immediately that

$$S(X, Y) = \frac{n+1}{2} g(X, Y) - 2g(AX, AY) .$$

Let $X = \Sigma X^\alpha e_\alpha + \Sigma X^{\alpha^*} Je_\alpha$. Then we have

$$S(X, X) = \frac{n+1}{2} g(X, X) - 2\Sigma\lambda_\alpha^2(X^\alpha X^\alpha + X^{\alpha^*}X^{\alpha^*}) . \tag{3}$$

Let $H(e_\alpha)$ be the holomorphic sectional curvature determined by e_α, $\alpha = 1, \cdots, n$. Then we have

$$H(e_\alpha) = g(R(e_\alpha, Je_\alpha)Je_\alpha, e_\alpha) = 1 - 2\lambda_\alpha^2 .$$

Since every holomorphic sectional curvature is greater than $1/2$, we have $\lambda_\alpha^2 < 1/4$, which, together with (3), implies $S(X, X) > \frac{n}{2} g(X, X)$. Thus $S - \frac{n}{2} g$

is positive definite so that $c_1(M) - \frac{n}{8\pi}[\Phi]$ and therefore $\frac{n-d+2}{8\pi}[\Phi] - \frac{n}{8\pi}[\Phi]$, in consequence of (1) and (2), are also positive definite. Hence we have $d < 2$, that is, $d = 1$, which completes the proof of Theorem 2.

The proof of Theorem 3 is quite similar to that of Theorem 2. In fact, since every holomorphic sectional curvature is positive, we have $\lambda_\alpha^2 < 1/2$, which, together with (3), implies $S(X, X) > \frac{n-1}{2} g(X, X)$. Thus $S - \frac{n-1}{2} g$ is positive definite so that $c_1(M) - \frac{n-1}{8\pi}[\Phi]$ and therefore $\frac{n-d+2}{8\pi}[\Phi] - \frac{n-1}{8\pi}[\Phi]$, in consequence of (1) and (2), are also positive definite. Hence we have $d < 3$, that is, $d = 1$ or 2.

Remark. From the proof of Theorem 2, we have the following result: *Let M be a compact complex hypersurface of the complex projective space $P_{n+1}(C)$. If every eigenvalue of the second fundamental form of M is in $(-1/2, 1/2)$, then M is a complex hyperplane $P_n(C)$.*

Bibliography

[1] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer, Berlin, 1966.
[2] B. Smyth, *Differential geometry of complex hypersurfaces*, Ann. of Math. **85** (1967) 246–266.

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